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Extensions of a theorem of Cauchy–Liouville

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ABSTRACT

We deal with the equations $\Delta_p u + f(u) = 0$ and $\Delta_p u + (p-1)g(u)|\nabla u|^p + f(u) = 0$ in $\mathbb{R}^N$, where $g(t)$ is a continuous function in $(0,\infty)$, $p > 1$ and $f(t)$ is a smooth function for $t > 0$. Under appropriate conditions on g and f we show that the corresponding equation cannot have nontrivial non-negative entire solutions.

1. Introduction

Liouville's Theorem for harmonic functions states that any positive harmonic function on $\mathbb{R}^N$ is a constant. An elementary proof of this classical result that depends on the mean-value property of harmonic functions can be found in [1]. It turns out that this property of positive entire harmonic functions is shared by solutions of more general elliptic equations in $\mathbb{R}^N$ or generally on Riemannian manifolds (see [7,10]).

Of particular interest is the work of Gidas and Spruck [9] where non-negative solutions of

$$\Delta u + u^{q-1} = 0$$

in $\mathbb{R}^N$ are shown to be trivial, provided that $N > 2$ and $2 \leqslant q < 2N/(N-2)$. It is known that this conclusion fails if $q \geqslant 2N/(N-2)$. For an extensive and interesting account on the problem and its history we refer the reader to the paper [13]. In this interesting paper, J. Serrin and H. Zou generalized the results in [9] considerably as follows. Consider a degenerate quasilinear elliptic equation of the form

$$\Delta_p u + f(u) = 0, \quad \text{in } \mathbb{R}^N, \ p > 1, \tag{1.1}$$

where f is a differentiable non-linearity and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so called p-Laplace operator for $p > 1$. The result of J. Serrin and H. Zou concerns non-negative solutions of (1.1) when $1 < p < N$ and the non-linearity f is subcritical. Recall that $f(t) \geqslant 0$ is subcritical if there is $1 < \alpha < Np/(N-p)$ such that

$$f'(t) \leqslant (\alpha - 1)\frac{f(t)}{t} \quad \forall t > 0. \tag{1.2}$$

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Theorem II(c) of [13] states that

If $1 < p < N$, $f \geqslant 0$ *is subcritical and there is* $q > p$ *such that*

$$f(t) \geqslant t^{q-1} \quad \textit{for } t \textit{ large}, \tag{1.3}$$

then (1.1) *has only the solution* $u \equiv 0$. *The same conclusion holds if* $q \in (1, p]$, $p < N$ *and* $\alpha \leqslant p$ *in* (1.2).

This generalizes the result of Gidas and Spruck mentioned earlier. Similar problems have been discussed in [2–6]. In a recent paper [8], the nonexistence of entire non-negative solutions of the degenerate elliptic inequality

$$Lu \geqslant h\big(|x|\big) f(u)$$

has been discussed. Here the operator L is quite general and includes the p-Laplacian. The functions h and f appearing in the inequality above are positive.

In a recent paper [11], McCoy considers problem (1.1) for $p = 2$. In [11], it is shown that if f is differentiable and satisfies, for all $t > 0$, the inequality

$$f'(t) \leqslant \frac{N+1}{N-1} \frac{f(t)}{t},$$

then any positive solution of the equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \ N \geqslant 2, \tag{1.4}$$

must be a constant. As a consequence, if $f(t) = 0$ has no positive root, then (1.4) has no positive solution. Actually, in [11] the above problem is discussed in a framework of a general Riemann manifold, and also for equations having some quadratic gradient term. We should point out here that there is no sign restriction on f.

Our purpose in this paper is to generalize the above result of McCoy to problem (1.1). In Section 2 we investigate Eq. (1.1) with f not necessarily of constant sign. More specifically we show that if $N \geqslant 2$ and $f(t)$ is a differentiable function satisfying

$$f'(t) \leqslant (p-1) \frac{N+1}{N-1} \frac{f(t)}{t}, \quad t > 0, \tag{1.5}$$

then any entire positive solution to (1.1) must be a constant. We observe that condition (1.5) for $f(t) \geqslant 0$ and $1 < p < N$ is stronger than condition (1.2), but we do not require condition (1.3) and we make no restriction on the sign of $f(t)$ or size of $p > 1$. For instance our result shows that the only positive entire solution of (1.1) with $f(t) = t^a - t^c$ where $0 \leqslant a \leqslant (p-1)(N+1)/(N-1) \leqslant c$ is $u(x) \equiv 1$.

In Section 3 of the present paper we discuss the equation

$$\Delta_p u + (p-1) g(u) |\nabla u|^p + f(u) = 0, \tag{1.6}$$

where $N \geqslant 2$ and $g(t)$ is a continuous function in $(0, \infty)$. We show that if $f(t)$ is a differentiable function (not necessarily of constant sign) satisfying a special growth condition depending on N and g, then any entire positive solution to (1.6) must be a constant.

2. *p*-Laplace equations

In this section we investigate the p-Laplace equation (1.1) rewritten as

$$|\nabla u|^{p-2}\big[\Delta u + (p-2) u_{ij} u_i u_j |\nabla u|^{-2}\big] + f(u) = 0, \quad p > 1, \tag{2.1}$$

where $u_i = u_{x_i}$, and the summation convention from 1 to N over repeated indices is in effect.

We need the following result:

Lemma 1. *Let* p *be a real number and* $N \geqslant 2$. *If* $u(x)$ *is a* C^2 *function and* u_i *denotes partial differentiation with respect to* x_i *then*

$$(p-1)u_{11}^2 + \sum_{i=2}^N u_{ii}^2 \geqslant \frac{(p-1)(N-1)+1}{N-1} u_{11}^2 - \frac{2}{N-1} \Delta u u_{11} + \frac{1}{N-1} (\Delta u)^2.$$

Proof. We have

$$\sum_{i=2}^N u_{ii}^2 \geqslant \frac{1}{N-1} \left(\sum_{i=2}^N u_{ii} \right)^2 = \frac{1}{N-1} (\Delta u - u_{11})^2.$$

Adding $(p-1)u_{11}^2$ to both sides of this inequality, after easy manipulations we get the desired inequality. □

Theorem 1. *Let $u(x) > 0$ satisfy Eq.* (2.1) *in $\mathbb{R}^N$, $N \geqslant 2$. If the function $f(t)$ is differentiable and satisfies*

$$f'(t) \leqslant (p-1)\frac{N+1}{N-1}\frac{f(t)}{t} \quad \forall t > 0,$$

then $u(x)$ must be a constant.

Proof. Let

$$\Phi = |\nabla u|^2 u^{-2}.$$

Consider a point x^* where $|\nabla u| > 0$. We know that the function $u(x)$ is smooth in $\{x \in \Omega\colon |\nabla u| > 0\}$ [14]. For $i = 1, \dots, N$ we find

$$\Phi_i = 2u_{ij}u_j u^{-2} - 2|\nabla u|^2 u_i u^{-3}, \tag{2.2}$$

and

$$\Phi_{ii} = 2u_{iij}u_j u^{-2} + 2u_{ij}u_{ij}u^{-2} - 8u_{ij}u_i u_j u^{-3} - 2|\nabla u|^2 u_{ii}u^{-3} + 6|\nabla u|^2 u_i u_i u^{-4}.$$

Since Eq. (2.1) is invariant under rotations and translations, we choose the coordinates so that, at the point x^* we have

$$|\nabla u| = u_1, \qquad u_i = 0, \quad i = 2, \dots, N. \tag{2.3}$$

Then we find

$$\Phi_{11} = 2u_{111}u_1 u^{-2} + 2u_{1j}u_{1j}u^{-2} - 10u_{11}u_1^2 u^{-3} + 6u_1^4 u^{-4},$$

and

$$\Delta\Phi = 2(\Delta u)_1 u_1 u^{-2} + 2u_{ij}u_{ij}u^{-2} - 8u_{11}u_1^2 u^{-3} - 2u_1^2 \Delta u u^{-3} + 6u_1^4 u^{-4}.$$

It follows that

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} = {} & 2u_1 u^{-2}\big[(\Delta u)_1 + (p-2)u_{111}\big] + 2u^{-2}\big[u_{ij}u_{ij} + (p-2)u_{1j}u_{1j}\big] \\ & - 2u_1^2 u^{-3}\big[\Delta u + (5p-6)u_{11}\big] + 6(p-1)u_1^4 u^{-4}.\end{aligned} \tag{2.4}$$

If we differentiate Eq. (2.1) with respect to x_1 and evaluate the result at the point x^* satisfying (2.3) we find

$$u_1^{p-2}\left[(\Delta u)_1 + (p-2)u_{111} + 2(p-2)\sum_{j=2}^N u_{1j}^2 u_1^{-1}\right] + (p-2)u_{11}u_1^{p-3}\big[\Delta u + (p-2)u_{11}\big] + f'u_1 = 0.$$

Since Eq. (2.1) at the point x^* reads as

$$u_1^{p-2}\big[\Delta u + (p-2)u_{11}\big] = -f,$$

we get

$$(\Delta u)_1 + (p-2)u_{111} = 2(2-p)\sum_{j=2}^N u_{1j}^2 u_1^{-1} + (p-2)u_{11} f u_1^{1-p} - f' u_1^{3-p}. \tag{2.5}$$

On the other hand, using Lemma 1 we find

$$\begin{aligned}u_{ij}u_{ij} + (p-2)u_{1j}u_{1j} &\geqslant (p-1)u_{11}^2 + \sum_{i=2}^N u_{ii}^2 + p\sum_{j=2}^N u_{1j}^2 \\ &\geqslant \frac{(p-1)(N-1)+1}{N-1}u_{11}^2 - \frac{2}{N-1}\Delta u\, u_{11} + \frac{1}{N-1}(\Delta u)^2 + p\sum_{j=2}^N u_{1j}^2.\end{aligned} \tag{2.6}$$

Inserting (2.5) and (2.6) into (2.4) we find

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} \geqslant 2u^{-2}\Bigg[&(4-p)\sum_{j=2}^{N} u_{1j}^2 + (p-2)u_{11} f u_1^{2-p} - f' u_1^{4-p}\\ &+ \frac{(p-1)(N-1)+1}{N-1}u_{11}^2 - \frac{2}{N-1}\Delta u u_{11} + \frac{1}{N-1}(\Delta u)^2\\ &- u_1^2 u^{-1}\big[\Delta u + (5p-6)u_{11}\big] + 3(p-1)u_1^4 u^{-2}\Bigg].\end{aligned} \tag{2.7}$$

Recall that we are assuming (2.3). From (2.2) we see that, for $i = 2, \dots, N$,

$$2u_{i1}u_1 u^{-2} = \Phi_i,$$

and therefore

$$4u_1^2 u^{-4}\sum_{i=2}^{N} u_{i1}^2 = \sum_{i=2}^{N}\Phi_i^2 \leqslant |\nabla\Phi|^2.$$

Consequently we have the inequality

$$2(4-p)u^{-2}\sum_{j=2}^{N} u_{1j}^2 \geqslant -|4-p|\frac{|\nabla\Phi|^2}{2\Phi}.$$

Using the latter estimate and the equation

$$-\Delta u = (p-2)u_{11} + f u_1^{2-p}$$

in (2.7) we find

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} &\geqslant -|4-p|\frac{|\nabla\Phi|^2}{2\Phi} + 2u^{-2}\Bigg[(p-2)u_{11} f u_1^{2-p} - f' u_1^{4-p} + \frac{(p-1)(N-1)+1}{N-1}u_{11}^2\\ &\quad + \frac{2}{N-1}\big[(p-2)u_{11} + f u_1^{2-p}\big]u_{11} + \frac{1}{N-1}\big[(p-2)u_{11} + f u_1^{2-p}\big]^2\\ &\quad - u_1^2 u^{-1}\big[4(p-1)u_{11} - f u_1^{2-p}\big] + 3(p-1)u_1^4 u^{-2}\Bigg]\\ &= -|4-p|\frac{|\nabla\Phi|^2}{2\Phi} + 2u^{-2}\Bigg[(p-2)u_{11} f u_1^{2-p} - f' u_1^{4-p}\\ &\quad + \frac{(p-1)(N-1)+1+2(p-2)+(p-2)^2}{N-1}u_{11}^2 + \frac{2}{N-1} f u_1^{2-p}u_{11} + \frac{2(p-2)}{N-1} f u_1^{2-p}u_{11}\\ &\quad + \frac{1}{N-1}\big(f u_1^{2-p}\big)^2 - u_1^2 u^{-1}\big[4(p-1)u_{11} - f u_1^{2-p}\big] + 3(p-1)u_1^4 u^{-2}\Bigg].\end{aligned}$$

Since

$$\frac{(p-1)(N-1)+1+2(p-2)+(p-2)^2}{N-1} = \frac{(p-1)(p+N-2)}{N-1}$$

and

$$\frac{2}{N-1} + \frac{2(p-2)}{N-1} = \frac{2(p-1)}{N-1}$$

we have

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} \geqslant -|4-p|\frac{|\nabla\Phi|^2}{2\Phi} + 2u^{-2}\Bigg[&-f' u_1^{4-p} + \frac{(p-1)(p+N-2)}{N-1}u_{11}^2\\ &+ \left(p-2+\frac{2(p-1)}{N-1}\right) f u_1^{2-p}u_{11} + \frac{1}{N-1}\big(f u_1^{2-p}\big)^2\\ &- u_1^2 u^{-1}\big[4(p-1)u_{11} - f u_1^{2-p}\big] + 3(p-1)u_1^4 u^{-2}\Bigg].\end{aligned} \tag{2.8}$$

From (2.2) we find

$$\Phi_1 = 2u_1 u_{11} u^{-2} - 2u_1^3 u^{-3},$$

whence

$$u_{11} = \frac{\Phi_1 u^2}{2u_1} + \frac{u_1^2}{u}.$$

Inserting this equality into (2.8) we get

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} \geqslant{}& -|4-p|\frac{|\nabla\Phi|^2}{2\Phi} + 2u^{-2}\biggl[-f' u_1^{4-p} + \frac{(p-1)(p+N-2)}{N-1}\left(\frac{\Phi_1 u^2}{2u_1} + \frac{u_1^2}{u}\right)^2 \\ &+ \left(p-2+\frac{2(p-1)}{N-1}\right)\left(\frac{\Phi_1 u^2}{2u_1} + \frac{u_1^2}{u}\right) f u_1^{2-p} + \frac{1}{N-1}\left(f u_1^{2-p}\right)^2 \\ &- \frac{u_1^2}{u}\left[4(p-1)\left(\frac{\Phi_1 u^2}{2u_1} + \frac{u_1^2}{u}\right) - f u_1^{2-p}\right] + 3(p-1)u_1^4 u^{-2}\biggr].\end{aligned}$$

Using the restriction on f we have

$$-f' u_1^{4-p} + \left(p-2+\frac{2(p-1)}{N-1}\right) u_1^2 \frac{f}{u} u_1^{2-p} + u_1^2 \frac{f}{u} u_1^{2-p} = u_1^{4-p}\left[-f'(u) + (p-1)\frac{N+1}{N-1}\frac{f(u)}{u}\right] \geqslant 0.$$

We also note that

$$\frac{(p-1)(p+N-2)}{N-1}\left(\frac{\Phi_1 u^2}{2u_1}\right)^2 \geqslant 0.$$

Therefore,

$$\begin{aligned}\Delta\Phi + (p-2)\Phi_{11} \geqslant{}& -|4-p|\frac{|\nabla\Phi|^2}{2\Phi} + \frac{(p-1)(p+N-2)}{N-1}\left(2\frac{\Phi_1 u_1}{u} + 2\frac{u_1^4}{u^4}\right) + \left(p-2+\frac{2(p-1)}{N-1}\right)\Phi_1 f u_1^{1-p} \\ &+ \frac{2}{N-1}\frac{(f u_1^{2-p})^2}{u^2} - 4(p-1)\frac{\Phi_1 u_1}{u} - 2(p-1)\frac{u_1^4}{u^4}.\end{aligned}$$

Since

$$\frac{2(p-1)(p+N-2)}{N-1} - 4(p-1) = \frac{2(p-1)(p-N)}{N-1},$$

and

$$\frac{2(p-1)(p+N-2)}{N-1} - 2(p-1) = \frac{2(p-1)^2}{N-1},$$

we find

$$\begin{aligned}\frac{\Delta\Phi + (p-2)\Phi_{11}}{\Phi} \geqslant{}& -|4-p|\frac{|\nabla\Phi|^2}{2\Phi^2} + \frac{2(p-1)(p-N)}{N-1}\frac{\Phi_1 u_1}{\Phi u} + \left(p-2+\frac{2(p-1)}{N-1}\right)\frac{\Phi_1 f u_1^{1-p}}{\Phi} \\ &+ \frac{2}{N-1}\left(f u_1^{1-p}\right)^2 + \frac{2(p-1)^2}{N-1}\frac{u_1^2}{u^2}.\end{aligned} \tag{2.9}$$

Now we look for an inequality complementary to (2.9). For $x_0 \in R^N$ fixed we define

$$J(x) = \left(a^2 - r^2\right)^2 \Phi,$$

where a is a positive constant and $r = |x - x_0|$. The function J is non-negative in the ball B centered at x_0 and radius a, and vanishes for $|x - x_0| = a$; therefore it must attain a maximum value at some (interior) point $x^* \in B$. We can assume that $|\nabla u| > 0$ in x^* (otherwise we have $\Phi \equiv 0$ in such a ball; if $\Phi \equiv 0$ in every ball then $\nabla u \equiv 0$ in R^N and the theorem follows). At x^* we have

$$J_i = -2\left(a^2 - r^2\right)\left(r^2\right)_i \Phi + \left(a^2 - r^2\right)^2 \Phi_i = 0. \tag{2.10}$$

Moreover, since the matrix $[\delta_{ij}+(p-2)u_iu_j|\nabla u|^{-2}]$ is positive definite when $p>1$, at x^* we have

$$\Delta J+(p-2)J_{ij}u_iu_j|\nabla u|^{-2}\leqslant 0.$$

Suppose that Eqs. (2.3) hold at x^*. Then

$$\begin{aligned}\Delta J+(p-2)J_{11}={}&2\left|\nabla r^2\right|^2\Phi-2\left(a^2-r^2\right)\Delta r^2\Phi-4\left(a^2-r^2\right)\nabla r^2\cdot\nabla\Phi+\left(a^2-r^2\right)^2\Delta\Phi\\&+(p-2)\left[2\left(\left(r^2\right)_1\right)^2\Phi-4\left(a^2-r^2\right)\Phi-4\left(a^2-r^2\right)\left(r^2\right)_1\Phi_1+\left(a^2-r^2\right)^2\Phi_{11}\right]\leqslant 0.\end{aligned}$$

This implies

$$\begin{aligned}\Delta\Phi+(p-2)\Phi_{11}\leqslant{}&-\frac{8r^2}{(a^2-r^2)^2}\Phi+\frac{4N}{a^2-r^2}\Phi+\frac{4}{a^2-r^2}\nabla r^2\cdot\nabla\Phi\\&+(p-2)\left[-\frac{2}{(a^2-r^2)^2}\left(\left(r^2\right)_1\right)^2\Phi+\frac{4}{a^2-r^2}\Phi+\frac{4}{a^2-r^2}\left(r^2\right)_1\Phi_1\right].\end{aligned}$$

By (2.10) we get

$$\Phi_1=2\frac{(r^2)_1\Phi}{a^2-r^2},\qquad\nabla\Phi=2\frac{\nabla r^2\Phi}{a^2-r^2}.\tag{2.11}$$

Therefore

$$\Delta\Phi+(p-2)\Phi_{11}\leqslant\frac{24r^2}{(a^2-r^2)^2}\Phi+\frac{4N}{a^2-r^2}\Phi+(p-2)\left[\frac{6}{(a^2-r^2)^2}\left(\left(r^2\right)_1\right)^2\Phi+\frac{4}{a^2-r^2}\Phi\right].$$

Since $((r^2)_1)\leqslant|\nabla r^2|=2r$ we find

$$\frac{\Delta\Phi+(p-2)\Phi_{11}}{\Phi}\leqslant\frac{24r^2}{(a^2-r^2)^2}+\frac{4N}{a^2-r^2}+|p-2|\left[\frac{24r^2}{(a^2-r^2)^2}+\frac{4}{a^2-r^2}\right].$$

Since $r^2<a^2$ we also find

$$\frac{\Delta\Phi+(p-2)\Phi_{11}}{\Phi}\leqslant\frac{Ca^2}{(a^2-r^2)^2},\quad C=24+4N+28|p-2|.$$

The latter estimate and (2.9) yield

$$\begin{aligned}\frac{Ca^2}{(a^2-r^2)^2}\geqslant{}&-|4-p|\frac{|\nabla\Phi|^2}{2\Phi^2}+\frac{2(p-1)(p-N)}{N-1}\frac{\Phi_1u_1}{\Phi u}+\left(p-2+\frac{2(p-1)}{N-1}\right)\frac{\Phi_1fu_1^{1-p}}{\Phi}\\&+\frac{2}{N-1}\left(fu_1^{1-p}\right)^2+\frac{2(p-1)^2}{N-1}\frac{u_1^2}{u^2}.\end{aligned}$$

By Eqs. (2.3) and (2.11) we find

$$\frac{\Phi_1u_1}{\Phi}=\frac{\nabla\Phi\cdot\nabla u}{\Phi}=2\frac{\nabla r^2\cdot\nabla u}{a^2-r^2}$$

and

$$\frac{|\nabla\Phi|}{\Phi}=\frac{2|\nabla(r^2)|}{a^2-r^2}=\frac{4r}{a^2-r^2}.$$

Therefore,

$$\begin{aligned}\frac{Ca^2}{(a^2-r^2)^2}\geqslant{}&-|4-p|\frac{8r^2}{(a^2-r^2)^2}+\frac{4(p-1)(p-N)}{N-1}\frac{\nabla r^2\cdot\nabla u}{(a^2-r^2)u}+2\left(p-2+\frac{2(p-1)}{N-1}\right)fu_1^{-p}\frac{\nabla r^2\cdot\nabla u}{a^2-r^2}\\&+\frac{2}{N-1}\left(fu_1^{1-p}\right)^2+\frac{2(p-1)^2}{N-1}\frac{|\nabla u|^2}{u^2}.\end{aligned}\tag{2.12}$$

By classical inequalities we have

$$4(p-1)(p-N)\frac{\nabla r^2\cdot\nabla u}{(a^2-r^2)u}\geqslant-(p-1)^2\frac{|\nabla u|^2}{u^2}-4(p-N)^2\frac{4r^2}{(a^2-r^2)^2},$$

and

$$2\left(p-2+\frac{2(p-1)}{N-1}\right)fu_1^{-p}\frac{\nabla r^2\cdot\nabla u}{a^2-r^2}\geqslant -4\left(|p-2|+\frac{2(p-1)}{N-1}\right)|f|u_1^{1-p}\frac{r}{a^2-r^2}$$
$$\geqslant -\frac{2}{N-1}\left(fu_1^{1-p}\right)^2-\tilde{C}\frac{r^2}{(a^2-r^2)^2},$$

with

$$\tilde{C}=\frac{2}{N-1}\left[(N-1)|p-2|+2(p-1)\right]^2.$$

Inserting these estimates into (2.12) we find

$$\frac{Ca^2}{(a^2-r^2)^2}\geqslant -|4-p|\frac{8r^2}{(a^2-r^2)^2}-\frac{4(p-N)^2}{N-1}\frac{4r^2}{(a^2-r^2)^2}-\tilde{C}\frac{r^2}{(a^2-r^2)^2}+\frac{(p-1)^2}{N-1}\frac{|\nabla u|^2}{u^2}.$$

The latter estimate implies the existence of a new constant $C=C_{N,p}$ such that

$$\frac{|\nabla u|^2}{u^2}\leqslant\frac{C_{N,p}a^2}{(a^2-r^2)^2}.$$

Hence, at the point x^* we have

$$J\left(x^*\right)=\frac{|\nabla u|^2}{u^2}\left(a^2-r^2\right)^2\leqslant C_{N,p}a^2.$$

But x^* is a point of maximum for $J(x)$ in B, therefore we also have

$$J(x_0)=\frac{|\nabla u|^2}{u^2}a^4\leqslant C_{N,p}a^2.$$

It follows that, at $x=x_0$,

$$\frac{|\nabla u|^2}{u^2}\leqslant\frac{C_{N,p}}{a^2}.$$

Letting $a\to\infty$ we find that $\nabla u=0$ at x_0. Since x_0 is arbitrary, we must have $\nabla u=0$ in $\mathbb{R}^N$. The theorem follows. $\square$

3. Equations with a gradient term

Let $g\in C(0,\infty)$, and set

$$G(t)=\int_1^t g(s)\,ds,\quad t>0.$$

We define an auxiliary function H as follows.

(1) If

$$\int_0^1 e^{G(s)}\,ds<\infty, \tag{3.1}$$

we define H by

$$H(t)=\int_0^t e^{G(s)}\,ds,\quad t>0. \tag{3.2}$$

(2) If

$$\int_0^1 e^{G(s)}\,ds=\infty\quad\text{and}\quad\int_1^\infty e^{G(s)}\,ds<\infty, \tag{3.3}$$

we define H by

$$H(t)=\int_t^\infty e^{G(s)}\,ds,\quad t>0. \tag{3.4}$$

(3) If

$$\int_0^1 e^{G(s)}\,ds = \infty \quad \text{and} \quad \int_1^\infty e^{G(s)}\,ds = \infty, \tag{3.5}$$

we choose a positive real number ℓ and define H in either of the following ways.

(a)

$$H(t) := \int_\ell^t e^{G(s)}\,ds, \quad t \geqslant \ell, \tag{3.6}$$

(b)

$$H(t) := \int_t^\ell e^{G(s)}\,ds, \quad 0 < t \leqslant \ell. \tag{3.7}$$

To proceed further let us make note of the following. Let u be a C^2 function on R^N and let H be one of the functions just defined (or any C^2 function on an interval containing the range of u). Set $w = H(u)$. Then a computation shows that

$$\Delta_p w = \left|H'(u)\right|^{p-2} H'(u)\Delta_p u + (p-1)\left|H'(u)\right|^{p-2} H''(u)|\nabla u|^p. \tag{3.8}$$

We are now ready to state our next theorem.

Theorem 2. *Let $g \in C(0,\infty)$ and let $f \in C^1(0,\infty)$. Let u be a solution of the equation*

$$\Delta_p u + (p-1)g(u)|\nabla u|^p + f(u) = 0 \quad \text{in } \mathbb{R}^N,\ N \geqslant 2. \tag{3.9}$$

(1) *Suppose*
   (a) *g satisfies* (3.1), *H is defined by* (3.2) *and $u > 0$ or*
   (b) *g satisfies* (3.5), *H is defined by* (3.6) *and $u > \ell$.*
   Moreover, let f satisfy

$$f'(t) \leqslant (p-1)\left[-g(t) + \frac{N+1}{N-1}\frac{e^{G(t)}}{H(t)}\right] f(t) \quad \forall t \in I, \tag{3.10}$$

   where $I = (0,\infty)$ in case of (a) *or $I = (\ell,\infty)$ in case of* (b). *Then $u \equiv C$ for some root C of $f(t)$.*

(2) *Alternatively, suppose*
   (a) *g satisfies* (3.3), *H is defined by* (3.4) *and $u > 0$ or*
   (b) *g satisfies* (3.5), *H is defined by* (3.7) *and $0 < u < \ell$.*
   Moreover, let f satisfy

$$f'(t) \leqslant (p-1)\left[-g(t) - \frac{N+1}{N-1}\frac{e^{G(t)}}{H(t)}\right] f(t) \quad \forall t \in J, \tag{3.11}$$

   where $J = (0,\infty)$ in case of (a) *or $J = (0,\ell)$ in case of* (b). *Then $u \equiv C$ for some root C of $f(t)$.*

Proof. We follow a method similar to that used in [12]. Note that in all cases H is a monotonic function, and in the sequel we denote by h the inverse function of H. In all cases we set

$$w := H(u).$$

We start by proving case (1). If (3.1) holds and if H is defined as in (3.2), then we note that $u > 0$ implies $w > 0$ on R^N. If (3.5) holds and H is defined by (3.6) then $u > \ell$ implies $w > 0$. In any case we use (3.8) and Eq. (3.9) to compute

$$\begin{aligned}\Delta_p w &= \left|H'(u)\right|^{p-2} H'(u)\Delta_p u + (p-1)\left|H'(u)\right|^{p-2} H''(u)|\nabla u|^p \\ &= e^{(p-1)G(u)}\left[\Delta_p u + (p-1)g(u)|\nabla u|^p\right] \\ &= -e^{(p-1)G(u)} f(u) = -\tilde{f}(w),\end{aligned}$$

where $\tilde{f}(t) := f(h(t))e^{(p-1)G(h(t))}$, $t > 0$. Since $w > 0$, we can use Theorem 1. The condition

$$\tilde{f}'(s) \leqslant (p-1)\frac{N+1}{N-1}\frac{\tilde{f}(s)}{s}, \tag{3.12}$$

in terms of $f(t)$ becomes condition (3.10). Thus case (1) of the theorem follows.

Next we prove case (2) of the theorem. If (3.3) holds and H is defined as in (3.4) then $u > 0$ implies again $w > 0$. If (3.5) holds and H is defined by (3.7) then $0 < u < \ell$ implies $w > 0$. In any case we use (3.8) to compute $\Delta_p w$, and proceeding as in the above case we find that

$$\Delta_p w + \tilde{f}(w) = 0,$$

where $\tilde{f}(t) = -f(h(t))e^{(p-1)G(h(t))}$, $t > 0$. Since $w > 0$, we again use Theorem 1. Now, in terms of f (3.12) becomes (3.11), and hence case (2) of the theorem follows. $\square$

Examples.

1. Let $g(t) = -\eta t^{-1}$ with $0 < \eta < 1$. Then g satisfies (3.1). G and H are given by

$$G(t) = \log\bigl(t^{-\eta}\bigr), \qquad H(t) = \frac{1}{1-\eta}t^{1-\eta}.$$

The corresponding condition (3.10) becomes

$$f'(t) \leqslant (p-1)\frac{N+1-2\eta}{N-1}\frac{f(t)}{t}, \quad t > 0.$$

The latter condition in case of $p = 2$ has been found in [11].

2. Let $g(t) = -\eta t^{-1}$ with $\eta > 1$. Then g satisfies (3.3). G and H are given by

$$G(t) = \log\bigl(t^{-\eta}\bigr), \qquad H(t) = \frac{1}{\eta-1}t^{1-\eta}.$$

The corresponding condition (3.11) becomes

$$f'(t) \leqslant (p-1)\frac{N+1-2\eta}{N-1}\frac{f(t)}{t}, \quad t > 0.$$

Also the latter condition in case of $p = 2$ has been found in [11].

3. Let $g(t) = -t^{-1}$. Then g satisfies (3.5) with $G(t) = -\log t$. If H is defined by (3.6) we have $H(t) = \log\frac{t}{\ell}$, $t > \ell$; if H is defined by (3.7) then $H(t) = \log\frac{\ell}{t}$, $0 < t < \ell$. The corresponding condition for f becomes

$$f'(t) \leqslant (p-1)\left[1 + \frac{N+1}{N-1}\frac{1}{\log t}\right]\frac{f(t)}{t},$$

which must hold for $t > \ell$ or for $0 < t < \ell$, respectively.

As a corollary of the preceding theorem we find that if $u(x)$ is a positive solution to

$$\Delta_p u + (p-1)g(u)|\nabla u|^p = 0$$

in $\mathbb{R}^N$ then $u(x)$ must be a constant. If the condition $g \in C(0,\infty)$ fails to hold then we may have non-constant solutions. For example, the radial function $u(x) = e^{|x|^2}$ satisfies the equation

$$\Delta u - \frac{1}{u}\left(1 + \frac{N}{2\log u}\right)|\nabla u|^2 = 0.$$

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